The Scatter Factor in the Reliability Assessment of Aircraft Structures

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The lack of reliability of the analytical methods of prediction of fatigue lives of structural parts and complete structures requires the use of "scatter factors" to be applied to the results of full-scale fatigue tests of aircraft components and of structures under loading representative of conditions of service. These factors, so far, have been selected on the basis of subjective judgment based on engineering experience and therefore cannot be related to any specific level of reliability. An attempt is made to define the scatter factor in probabilistic terms as the ratio between the best estimator of the fatigue life obtained in a small number of tests and the time to the first failure in a fleet of aircraft of specified size, at a specified reliability level. It is shown that with this definition the scatter factor can be related to the result of a single full-scale test of a complete structure and associated with a specific reliability level. The effect of the minimum fatigue life on the scatter factor is studied and its significance demonstrated in a set of tables computed on the basis of the derived equations.

I. Introduction

CONVENTIONAL procedures of life estimation of aircraft involve the assumption of arbitrary scatter factors, usually in the range between 2 and 4. Applied to the "estimated" life of a structure based on development tests, scatter factors are supposed to provide the "safe" or "certifiable" life. The probability of occurrence of individual lives shorter than the "safe" life in the fleet of airplanes thus remains undefined but is, presumably, considered acceptable (or zero). Since a statement concerning the reliability level of the design cannot be made, no reliability level can be attached to the selected scatter factor, nor can any statistical significance be attached to the "estimated" life. In view of the unreliability of available conventional analytical methods of damage prediction, fatigue testing under loading sequences representative of expected operational conditions provides the only means of arriving at realistic life estimates. These estimates are, however, usually obtained from a single, full-scale test, its cost being in general too high for even a small number of replications.

Attempts to impose conventional procedures of reliability estimation developed for multicomponent, "maintainable" electronic and mechanical systems are obviously futile, since in such "maintainable" systems failure of components is an acceptable contingency. Failure is provided for by multiple standby redundancies or by "renewal" of parts before the expected failure time, which is estimated on the basis of the mean time to failure (MTF) of the component elements obtained in tests of sufficiently large replication. This is possible because of the low cost of both the elements and the testing procedure. The operational life or mean time to failure of easily maintainable ("renewable") multicomponent systems is computed on the basis of simple assumptions concerning element assemblies in the absence of the strong interactions characteristic of structural systems. The effect of scatter is provided for by the assurance of a design life (MTF of the system) that is a large multiple of the anticipated operational life. The computed probability of failure at the end of the operational life thus can be kept arbitrarily low, primarily by multiplying standby redundancies. In spite of the fact that the operational lives of most space systems are only of the order of weeks or months (rather than years), the obvious success of this procedure in the space effort had to be paid for by a drastic improvement in the level of quality control in the production of elements and complex systems. The economic consequences would hardly be bearable in production processes governed by considerations of economy and by limitations in the allocation of resources.

The full-scale fatigue test has more than one apparent purpose. Its crucial role in the identification of fatigue-critical details and in the determination of their fatigue life, as well as of the fatigue life of the whole structure on which its certification and economic utilization must be based, is the result of the general uncertainty implicit in all deterministic analytic procedures of fatigue design and prediction of fatigue life. This is true even under a uniquely defined load sequence, except in the simplest case of the pre-existence of a single, dominating, geometrically definable defect in a known stress field of constant amplitude and constant mean stress.[1] The assessment of scatter factors to be applied to the test lives of critical structural elements or locations as well as of the entire structure, with credibly quantifiable reliability levels, would thus appear more desirable than the subjective assessments by the designer, on the basis of his "engineering judgment," of scatter factors unrelated to a quantifiable reliability level. Even if the limit of the obtainable objective reliability level cannot attain fictitious levels of the order of $R \geq 0.99$ presumably attainable by subjective assessment, it may be interesting to note that studies of the validity of subjective assessment of the confidence in the prediction of the true value of certain events within the individual's frame of knowledge[2] have shown this confidence to be consistently overrated. The conclusion of these studies is that "people believe they have a much better picture of the truth than they really do." This is clearly reflected in the confidence of designers in their deterministic analytical procedures and in their judgment, as well as in the excessive demands made by them on the accuracy of quantitative reliability assessments, while never doubting the adequacy of their subjective assessment.

In structural systems, failure of a critical component or at a critical location is not a contingency but an emergency, to be guarded against by design and inspection by NDT methods. As a result of the strong interactions between components, failure of a single component may, directly or through a chain reaction, lead to failure of the system; the failure-arresting effect of structural redundancies is uncertain and their number obviously limited. The element interactions in the highly

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redundant structural systems also preclude the assessment of the life of the full-scale structure from that of the lives of smaller sized critical elements obtained from tests with moderate replication that is not economically prohibitive. This fact leaves the full-scale test as the only means of realistic life assessment.

An improvement in the conventional procedures of life prediction by subjective fixed scatter factors can be achieved through the application of the concept of the "expected time to first failure in a fleet."[3] This concept recognizes the fact that structural systems rely on strong, sequentially interacting redundancies involving large elements and are therefore not easily maintainable by "renewal" of elements. A credible value of the MTF can neither be determined by test replication of the whole system, because of excessive cost and time expenditure, nor can it be computed from component tests, because of the complex structural interactions and sequential redundancies. The MTF concept is, moreover, operationally irrelevant since, in the absence of "renewal" of elements, it implies that at that time about one-half of the fleet of airplanes has actually failed. The reliability of the fleet is defined as the probability that no fleet member will fail before having attained the certified or "safe" life. The certified ("safe") life is assumed, by implication, to represent the life up to which no failure in the fleet is anticipated; it becomes identical therefore, with a lower bound of the "time to first failure." By this definition it becomes not only a function of the anticipated fleet size but depends, moreover, on the type of structural damage that is expected to produce failure, and thus on the damage rate. The latter determines the shape of the distribution function of fatigue lives, the parameters of which, in turn, depend on material fatigue properties, structural configuration, manufacturing quality, and airplane usage.

Considering the statistical variability of all these effects, it does not seem possible to produce aircraft of identical safety or flight reliability, independent of fleet size. Identical safety of flight requirements are, in fact, usually imposed on the design and production process of (a fleet of) aircraft. However, the belief that a product can be obtained which is not subject to a statistical variation of a range that necessarily increases with increasing sample (fleet) size contradicts the basic rules of probabilistic modeling of the physical reality.

The result of the introduction of the "time to first failure" as the central parameter of the structural reliability analysis involves the replacement, in this analysis, of the methods of conventional statistical theory based on statistical population averages and ranges of the deviation from them, by the theory of statistics of extremes, which is a branch of order-statistics.[4] The identification of the "time to first failure" with the "certifiable life" of the fleet establishes the missing relation between this life and the probability of its being exceeded in the fleet, which is the fleet reliability. Identification of the "time to first failure" as an extremal statistic relates it to a specific shape of the associated distribution function. It also provides the basis for a simple and direct procedure of reliability demonstration in acceptance testing. Thus it becomes an easily enforceable part of the procurement acceptance procedures instead of being, as at present, a source of ambiguous or arbitrary decisions, usually arrived at by a compromise between conflicting "expert" opinion.

Research studies have been undertaken under AFML sponsorship, mainly by the Boeing Company[5,6] and by the McDonnell Douglas Corp.,[7] in which groups of available in-service operational data on full-scale aircraft structures as well as multiple test data on structural parts and material specimens have been statistically evaluated with the purpose of obtaining credible values of the parameters required for the evaluation of the relevant distribution functions. The results of those studies seem to justify the shift of concern from the (unknown) distribution of the population of lives in the whole fleet to the (extremal) distributions of lives of the weakest

Table 1 Representative values of the shape parameter α

	In the reliability range $0.8 < R < 0.9$	
	α	
Material	Short life	Long life
Aluminum	4.5	3.5
Titanium	3.0	2.5
Steel (100-200 ksi)	3.5	3.0
Steel (200-300 ksi)	2.5	2.0
	In the median range $R \sim 0.5$	
	α	
Material	Short life	Long life
Aluminum	8.0	6.0
Titanium	6.5	4.0
Steel (100-200 ksi)	7.0	5.0
Steel (200-300 ksi)	5.0	3.5

members of this fleet ("shortest lives"). They also justify the selection of the two-parameter Third Asymptotic Distribution of extreme (smallest) values, also known as the Weibull distribution, as physically germane to the purposes of reliability analysis based on extremal statistics. Such justification on physical grounds is far more credible than statistically demonstrated "fitting" of available multiple-test results. It provides the basis for extrapolation of these results beyond the practical testing probability range, without which life predictions and reliability assessments become pure fiction.[8]

Estimates of the shape parameters α of these distributions based on the data collected by the Boeing Company and evaluated under the assumption of zero minimum life (two-parameter distribution) suggest tentative figures for different types of structural aircraft metals and for two different classes of structures presented in Table 1. The figures represent averages obtained in tests of representative structural configuration and manufacturing quality characteristic of current long range (transport) and short range (fighter) aircraft. They are subject to significant statistical variation, the specified numbers representing the range of values in the quantile range $0.80 < R < 0.9$. They have been obtained in tests under load sequences representative of airplane usage in the two general classes of aircraft structures (long life and short life) considered. The probability of values of α smaller than those specified, implying larger scatter, is therefore between $0.10 < (1-R) < 0.20$.

This may not appear to be a very satisfactory state of "knowledge" of the parameter on which to base a reliability estimate; unfortunately the data from which the figures in Table 1 have been derived are, up to now, the only existing test data of sufficient replication on which a credible quantitative estimate of the parameter α can be based, within known limits of credibility defined by the observed scatter. Such scatter may, in the eyes of the statistician, produce a reliability assessment of undesirably low confidence level. It nevertheless provides at least the possibility of a rough but objective evaluation of the relation between reliability and scatter factor, based on time to the first failure in fleets of given size, of aircraft of different materials, of currently accepted manufacturing quality, and of usage representative of the assumptions underlying their design. It would be more desirable to be able to account specifically for the separate effects of material properties, manufacturing quality, structural configuration, and airplane to airplane usage. However, this can hardly be more than a pious wish, considering the time, expense, and effort involved in the procurement even of the limited data presented by Refs. 5-7. In view of the not insignificant scatter of the shape parameter indicated by those data, it is doubtful whether a higher reliability level than $R \sim 0.95$ could and should, in fact, be a design goal. The belief, frequently expressed by designers, that deterministic analytical techniques, such as fracture-mechanics based crack-propagation analysis, can predict the fatigue life of a structure more reliably, reflects a basic misunderstanding of the concept of reliability. It also reflects a reluctance to face the fact that "deterministic techniques" are based on the deliberate suppression of the scatter existing in all physical relations that underlie such techniques. It is not the purpose of the scatter factor to represent the trend of first-order influences on the mean, median, or modal fatigue life ("scale parameter") of a structure, but to account for the scatter implicit in the deterministic techniques by which this trend might be predicted. The "reliability" of that prediction is obviously unrelated to the statistical scatter; it refers only to the uncertainty arising from insufficient knowledge concerning the selection of methods and parameters of the deterministic techniques. Thus, for instance, the known shortcomings of the rule of linear damage accumulation in predicting (mean) fatigue life cannot be adequately dealt with in terms of lack of its "reliability;" it requires its replacement by a more effective technique for the prediction of the trend, such as full-scale testing.

II. The Scatter Factor

This section is concerned with the probabilistic interpretation of the scatter factor by introducing this factor itself as a new statistical variable while retaining the two-parameter form of the Weibull distribution. The effect of the introduction of the third parameter, which affects the existence of a nonzero minimum life,[9] on this scatter factor is dealt with in Sec. IV.

The scatter factor, even when conventionally defined as an empirically specified multiplier, is, though clearly a fiction, nevertheless a practical concept in the context of a workable simple procedure of life-prediction. Like the "safety factor" in civil engineering design practice, it has been accepted widely by aircraft designers and operators because of its comforting simplicity. The scatter factor interpreted as a statistical variable related to the time to first failure in a fleet, on the other hand, is a clearly and rationally defined theoretical concept. However, it confronts the structural designer, the testing engineer, and the operator with questions, the previously simple answers to which are disappearing, together with the conventional simple concept of the scatter factor. For what "life" and at what reliability level is the structure to be designed, and what is the design significance of the results of the necessarily small number of fatigue tests of critical structural parts and, in particular, of the (usually single) fatigue test of the full-scale structure?

An operationally useful definition of the scatter factor satisfies the requirements of the designer as well as the operator of a fleet of aircraft. The definition also establishes its relation to the results of a very small number of development tests of structural parts, or to the single test of the full-scale structure: It is the ratio between the point estimator $\hat{\beta}$ of the location parameter β of the distribution of the population of fatigue lives, and the time to first failure. While the distribution of the total population in unknown, the fact that the interest in this distribution centers on its central region (location parameter) makes the estimation of the location parameter rather insensitive to the selection of the form of the distribution. The logarithmic normal, the gamma, and the two-parameter Weibull distributions are practically indistinguishable from each other over this central region. Selecting, for the sake of the simplicity of the analysis, the two-parameter Weibull distribution with location (scale) parameter (characteristic value) β and presumably known shape parameter α (see Table 1) as the distribution of the population y of fatigue lives in the fleet, the definition of the scatter S becomes

$$S=\hat{\beta}/y_1 \tag{1}$$

where $\hat{\beta}$ is the maximum likelihood point estimator of the scale parameter β for a sample of size n and y_1 is the statistically variable time to first failure in a fleet of size m.

The point estimator $\hat{\beta}$ for n observations y_i (Ref. 4, p. 297) is

$$\hat{\beta}=\left[\frac{1}{n}\sum_{1}^{n} y_i^{\alpha}\right]^{1/\alpha} \tag{2}$$

provided α is known. The distribution function $f(\hat{\beta})$ can be obtained by introducing the new variable $z_i=(y_i/\beta)^{\alpha}$ and utilizing the known fact that the sum

$$2nW=2\sum_{1}^{n} z_i=2\sum (y_i/\beta)^{\alpha}=2n(\hat{\beta}/\beta)^{\alpha} \tag{3}$$

has a chi-square distribution with $2n$ degrees of freedom

$$f(W)\mathrm{d}W=[2^n\Gamma(n)]^{-1}W^{n-1}e^{-nW/2} \tag{4}$$

and therefore

$$f(\hat{\beta})\mathrm{d}\hat{\beta}=\frac{n^n}{\Gamma(n)}\frac{\alpha}{\beta}\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha n-1}\times\exp\left[-n\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}\right]\mathrm{d}\hat{\beta} \tag{5}$$

The distribution of the smallest value y_1 of the Weibull variate with parameters (β,α) in a sample size m is

$$f_1(y_1)=\frac{\alpha}{\beta_1}\left(\frac{y_1}{\beta_1}\right)^{\alpha-1}e^{-(y_1/\beta_1)^{\alpha}} \tag{6}$$

where the scale parameter of $f_1(y_1)$, $\beta_1=\beta/m^{1/\alpha}$ while the shape parameter is the same as for $f(y)$. Hence

$$f_1(y_1)=\frac{\alpha m}{\beta}\left(\frac{y_1}{\beta}\right)^{\alpha-1}\exp\left[-m\left(\frac{y_1}{\beta}\right)^{\alpha}\right] \tag{7}$$

The distribution of the quotient $S=\hat{\beta}/y_1$ is therefore

$$f(S)=\int_0^{\infty}f(\hat{\beta})\cdot f_1(y_1)y_1\mathrm{d}y=\int_0^{\infty}f(\hat{\beta})\cdot f_1(\hat{\beta}/S)(\hat{\beta}/S)\mathrm{d}(\hat{\beta}/S)$$

$$=\int_0^{\infty}(\hat{\beta}/S^2)f(\hat{\beta})f_1(\hat{\beta}/S)\mathrm{d}\hat{\beta}$$

$$=\frac{n^n m\alpha^2}{\Gamma(n)\beta\cdot S^{\alpha+1}}\int_0^{\infty}\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha(n+1)-1}$$

$$\times\exp\left[-\left(n+\frac{m}{S^{\alpha}}\right)\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}\right]\mathrm{d}\hat{\beta} \tag{8}$$

With the abbreviations

$$A=n+mS^{-\alpha},\quad y=A\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}$$

and

$$\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha}=\frac{u}{A},\quad \frac{\alpha}{\beta}\left(\frac{\hat{\beta}}{\beta}\right)^{\alpha-1}\mathrm{d}\hat{\beta}=\frac{\mathrm{d}u}{A}$$

the integral, Eq. (8), is transformed into

$$f(S)=\frac{n^n m\alpha^2}{\Gamma(n)S^{\alpha+1}}\int_0^{\infty}u^n e^{-u}\mathrm{d}u=\frac{\alpha m n^{n+1}S^{\alpha n-1}}{(m+nS^{\alpha})^{n+1}} \tag{9}$$

since $\int_0^{\infty}u^n e^{-u}\mathrm{d}u=n\Gamma(n)$.

By integration of the density function Eq. (9)

$$\int_0^{\infty}f(S)\mathrm{d}S=\left[\frac{nS^{\alpha}}{m+nS^{\alpha}}\right]^n=F(S) \tag{10}$$

the distribution function $F(S)$ is obtained. Since the reliability R is the probability of values $\leq$ it follows that

$$R=F(S)=\left[\frac{S^{\alpha}}{(m/n)+S^{\alpha}}\right]^{n}=[F'(S)]^{n} \tag{11}$$

where

$$F'(S)=\frac{S}{m/n+S^{\alpha}} \tag{12}$$

A degenerate form of Eqs. (9) and (10) with $m/n=1$ is obtained when instead of a fleet of size m a single member of the population $(m=1)$ is considered. This form has been obtained by Impellizzeri and co-workers[7] and erroneously designated as the distribution of the scatter factor. It is, however, not a "scatter factor" associated with a significant reliability level of the fleet, but represents the distribution of the ratio $(\hat{\beta}/y)$ between the point estimator of the scale parameter $\hat{\beta}$ for a sample of size n and *any* member of the fleet. The small values of S obtained in the reported investigation are therefore misleading; it is not the (small) sample of size n used to estimate the scale parameter β and its relation to *any* member of the fleet but the size m of the fleet that determines the fleet reliability by relating it to its weakest (shortest lived) member.

III. Evaluation of the Scatter Factor

Evaluation of Eq. (11) for different values of n has shown that the effect of the sample size n on the scatter factor S is surprisingly small. This is a conclusion of considerable practical significance, particularly with respect to the results of full-scale tests of structures or large parts; if the improvement in the estimator $\hat{\beta}$ that results from an increase in the sample size beyond $n=1$ is not significant, the considerable cost of replication of full-scale tests can hardly be justified. The estimate of the scatter factor and associated reliability can therefore be based on the results of a single test for the estimator $\hat{\beta}$. The results of comparative computations show that the error in the estimation of the scatter factor S by using $n=1$ instead of $n=2$ or 3 at the significant reliability levels $R \geq 0.75$ does not exceed 2%, and that even this small gain from test replication vanishes beyond $n=3$.

With $n=1$, Eqs. (9) and (11) take the form

$$f(S)=\frac{\alpha m S^{\alpha-1}}{(m+S^{\alpha})^{2}} \tag{9a}$$

and

$$F(S)=\frac{S^{\alpha}}{m+S^{\alpha}}=R \tag{11a}$$

Either the mode or median can be specified as the location parameter. Differentiating Eq. (9a) with respect to S, the mode $\tilde{S}$ is obtained from $\mathrm{d}f(S)/\mathrm{d}S=0$

$$\tilde{S}=(m)^{1/\alpha}\left(\frac{\alpha-1/n}{\alpha+1}\right)^{1/\alpha} \tag{13}$$

while the median

$$\mathbb{S}=(m)^{1/\alpha}\left[\frac{1}{n(2^{1/n}-1)}\right]^{1/\alpha} \tag{14}$$

For $n=1$ therefore

$$\tilde{S}=(m)^{1/\alpha}\left(\frac{\alpha-1}{\alpha+1}\right)^{1/\alpha} \tag{13a}$$

and

$$\mathbb{S}=(m)^{1/\alpha}>\tilde{S} \tag{14a}$$

with the quantiles $R(\tilde{S})=\alpha-1/2\alpha$ and $R(\mathbb{S})=0.5$; with increasing value of α the location of the mode of the distribution $f(S)$ for $n=1$, which is at $R=0.25$ for $\alpha=2$, approaches rapidly that of the median $(R=0.5)$.

On the basis of Eqs. (13a) and (14a) the inadequacy of the definition of the scatter factor as the ratio $(\hat{\beta}/y)$ can be illustrated by using the fact that the two definitions coincide for $m \doteq 1$ (single-member "fleet"). The median ratio of the point-estimator $\hat{\beta}$ and *any* member of the population y is therefore $\mathbb{S}=1$ (instead of $m^{1/\alpha}$), which results in a substantial underestimate of the true scatter factor.

Expressing Eq. (11a) in terms of the reduced variable $(S/\mathbb{S})=s$, the normalized form of Eq. (11a)

$$F(s)=\frac{s^{\alpha}}{1+s^{\alpha}}=R \tag{15}$$

is obtained.

This Pareto-type form has been used in econometrics on a purely heuristic basis for the representation of the distribution of incomes and is known as the Champernowne distribution.[10] Being related to the specific model represented by the definition of the scatter factor, Eq. (1), it is of relevance in all situations in which reliability and risk assessment for a finite population have to be based on a small sample estimate of its location parameter.

The relation between scatter factor and reliability is obtained by solving Eq. (11) for S.

$$S=(m/n)^{1/\alpha}\left[\frac{R^{1/n}}{1-R^{1/n}}\right]^{1/\alpha} \tag{16}$$

and for $n=1$

$$S=S_{1}=m^{1/\alpha}\left(\frac{R}{1-R}\right)^{1/\alpha} \tag{17}$$

or

$$S_{1}/\mathbb{S}=s_{1}=\left(\frac{R}{1-R}\right)^{1/\alpha} \tag{18}$$

The ratio (S/S_1) illustrates the effect of the sample size n used in estimating $\hat{\beta}$

$$S/S_{1}=(1/n)^{1/\alpha}\left[R^{(1/n-1)}\frac{1-R}{1-R^{1/n}}\right]^{1/\alpha} \tag{19}$$

Equations (16) and (17) have been evaluated for different combinations of m, n, α, and R. The results presented in Tables 2-7 for $\epsilon=0$ (zero minimum life) support the conclusion that the scatter factor S_1, when applied to the result of a single full-size fatigue test $(n=1)$, predicts with reasonable accuracy the "safe life" at a specified reliability level.

IV. Effect of Minimum Fatigue Life on the Scatter Factor

The "minimum life" is a characteristic feature of the fatigue process.[1] It reflects the fact that fatigue failure under repeated load amplitudes of intensities that produce no visible signs of deterioration of the structural resistance on the first or several applications is caused by progressive damage processes in the structural metal that require a substantial number of repetitions and reversals before external traces (cracks) can be made visible under the optical magnifications used in NDT. There is thus a distinct difference in the failure mechanisms leading to failure on first loading or failure under a relatively small number of load repetitions preceded by extensive plastic deformation (cyclic plastic stressing) and failure under a substantial number of load repetitions within a nominally "elastic" range (fatigue). The assumption that the distribution of fatigue lives can have a zero lower limit con-

Table 2 Scatter factors S_n for minimum life ratios $0 \le \epsilon \le 0.10$ and fleet sizes $3 \le m \le 1000$

		Reliability level $R=0.5$, $n=1$				
$m=3$,	$\alpha=$		2	3	4	5
	$\epsilon=$	0.00	1.7	1.4	1.3	1.3
		0.01	1.7	1.4	1.3	1.2
		0.05	1.7	1.4	1.3	1.2
		0.10	1.6	1.4	1.3	1.2
$m=25$,	$\epsilon=$	0.00	5.0	2.9	2.2	1.9
		0.01	4.8	2.9	2.2	1.9
		0.05	4.2	2.7	2.1	1.8
		0.10	3.6	2.5	2.0	1.8
$m=100$,	$\epsilon=$	0.00	10.0	4.6	3.2	2.5
		0.01	9.2	4.5	3.1	2.5
		0.05	6.9	3.9	2.9	2.3
		0.10	5.3	3.4	2.6	2.2
$m=250$,	$\epsilon=$	0.00	15.8	6.3	4.0	3.0
		0.01	13.8	6.0	3.9	3.0
		0.05	9.1	5.0	3.5	2.7
		0.10	6.4	4.1	3.1	2.5
$m=1000$,	$\epsilon=$	0.00	31.6	10.0	5.6	4.0
		0.01	24.2	9.2	5.4	3.9
		0.05	12.5	6.9	4.6	3.5
		0.10	7.8	5.3	3.9	3.1

Table 3 Scatter factors S_n for minimum life ratios $0 \le \epsilon \le 0.1$ and fleet sizes $3 \le m \le 1000$

		Reliability level $R=0.5$, $n=3$				
$m=3$,	$\alpha=$		2	3	4	5
	$\epsilon=$	0.00	1.9	1.5	1.4	1.3
		0.01	1.9	1.5	1.4	1.3
		0.05	1.9	1.5	1.4	1.3
		0.10	1.8	1.5	1.3	1.3
$m=25$,	$\epsilon=$	0.00	5.6	3.2	2.3	2.0
		0.01	5.4	2.1	2.3	2.0
		0.05	4.6	2.9	2.2	1.9
		0.10	3.9	2.6	2.1	1.8
$m=100$,	$\epsilon=$	0.00	11.3	5.0	3.4	2.6
		0.01	10.3	4.8	3.3	2.6
		0.05	7.5	4.2	3.0	2.4
		0.10	5.6	3.6	2.7	2.3
$m=250$,	$\epsilon=$	0.00	18.0	6.9	4.2	3.2
		0.01	15.3	6.5	4.1	3.1
		0.05	9.7	5.3	3.6	2.9
		0.10	6.7	4.3	3.2	2.6
$m=1000$,	$\epsilon=$	0.00	36.8	10.9	5.9	4.2
		0.01	26.6	10.0	5.7	4.1
		0.05	13.1	7.3	4.8	3.6
		0.10	8.0	5.5	4.0	3.2

Table 4 Scatter factors S_n for minimum life ratios $0 \le \epsilon \le 0.1$ and fleet sizes $3 \le m \le 1000$

		Reliability level $R=0.90$, $n=1$				
$m=3$,	$\alpha=$		2	3	4	5
	$\epsilon=$	0.00	5.2	3.0	2.3	1.9
		0.01	5.0	2.9	2.3	1.9
		0.05	4.3	2.7	2.1	1.8
		0.10	3.7	2.5	2.0	1.8
$m=25$,	$\epsilon=$	0.00	15.0	6.1	3.9	3.0
		0.01	13.1	5.8	3.8	2.9
		0.05	8.8	4.9	3.4	2.7
		0.10	6.3	4.0	3.0	2.5
$m=100$,	$\epsilon=$	0.00	30.0	9.7	5.5	3.9
		0.01	23.2	8.9	5.2	3.8
		0.05	12.2	6.7	4.5	3.4
		0.10	7.7	5.2	3.8	3.0
$m=250$,	$\epsilon=$	0.00	47.4	13.1	6.9	4.7
		0.01	32.4	11.7	6.5	4.5
		0.05	14.3	8.2	5.3	4.0
		0.10	8.4	5.9	4.3	3.4
$m=1000$,	$\epsilon=$	0.00	94.9	20.8	9.7	6.2
		0.01	48.9	17.3	9.0	5.9
		0.05	16.7	10.5	6.8	4.9
		0.10	9.1	7.0	5.2	4.1

Table 5 Scatter factors S_n for minimum life ratios $0 \le \epsilon \le 0.1$ and fleet sizes $3 \le m \le 1000$

		Reliability level $R=0.90$, $n=3$				
$m=3$,	$\alpha=$		2	3	4	5
	$\epsilon=$	0.00	5.3	3.0	2.3	1.9
		0.01	5.1	3.0	2.3	1.9
		0.05	4.4	2.8	2.2	1.9
		0.10	3.7	2.8	2.2	1.9
		0.10	3.7	2.5	2.0	1.8
$m=25$,	$\epsilon=$	0.00	15.3	6.2	3.9	3.0
		0.01	13.4	5.9	3.8	2.9
		0.05	8.9	4.9	3.4	2.7
		0.10	6.3	4.1	3.0	2.5
$m=100$,	$\epsilon=$	0.00	30.6	9.8	5.6	3.9
		0.01	23.6	9.0	5.3	3.8
		0.05	12.3	7.0	4.5	3.4
		0.10	7.7	5.2	3.8	3.0
$m=250$,	$\epsilon=$	0.00	48.3	13.2	7.0	4.8
		0.01	32.8	11.8	6.6	4.6
		0.05	14.4	8.2	5.4	4.0
		0.10	8.4	6.0	4.4	3.4
$m=1000$,	$\epsilon=$	0.00	97.1	21.0	9.8	6.3
		0.01	49.3	17.5	9.0	5.9
		0.05	16.7	10.5	6.8	4.9
		0.10	9.1	7.0	5.2	4.1

tradicts therefore the physical reality which requires the consideration of a "minimum life." This is reflected by the introduction of a third parameter $y_{\min} = \omega > 0$ into the (extremal) distribution function $P(y)$.

The principal, well-known difficulty in the use of the three-parameter distribution function

$$-\ln[1-P(y)] = \left[\frac{y-\omega}{\beta-\omega}\right]^{\alpha} \tag{20}$$

valid for $y > \omega > 0$ is the interrelation between the three parameters α, β, and ω, which precludes their independent estimation. A first estimate of the minimum life can be based on a visual inspection of the "linearity" of the plot of Eq. (20) on extreme value probability paper with the aid of its transformation into the straight-line relation[9]

$$\ln(y-\omega) = \ln(\beta-\omega) + z/\alpha \tag{21}$$

where

$$z = \ln\{-\ln[1-P(y)]\}$$

Table 6 Scatter factors S_n for minimum life ratios $0 \leq \epsilon \leq 0.1$ and fleet sizes $3 \leq m \leq 1000$

Reliability level $R=0.99$, $n=1$						
$m=3$,	$\alpha=$		2	3	4	5
	$\epsilon=$	0.00	17.2	6.7	4.2	3.1
		0.01	14.8	6.3	2.0	3.1
		0.05	9.5	5.2	3.6	2.8
		0.10	6.6	4.3	3.2	2.6
$m=25$,	$\epsilon=$	0.00	49.8	13.5	7.1	4.8
		0.01	33.4	12.0	6.7	4.6
		0.05	14.5	8.3	5.4	4.0
		0.10	8.5	6.0	4.4	3.5
$m=100$,	$\epsilon=$	0.00	99.5	21.5	10.0	6.3
		0.01	50.1	17.8	9.2	6.0
		0.05	16.8	10.6	6.9	5.0
		0.10	9.2	7.0	5.3	4.1
$m=250$,	$\epsilon=$	0.00	157	29.1	12.5	7.6
		0.01	61.4	22.7	11.2	7.1
		0.05	17.8	12.1	8.0	5.7
		0.10	9.5	7.6	5.8	4.6
$m=1000$,	$\epsilon=$	0.00	314	46.3	17.7	10.0
		0.01	76.1	31.8	15.2	9.2
		0.05	18.9	14.1	9.7	6.9
		0.10	9.7	8.4	6.6	5.3

Table 7 Scatter factors S_n for minimum life ratios $0 \leq \epsilon \leq 0.1$ and fleet sizes $3 \leq m \leq 1000$

Reliability level $R=0.99$, $n=3$						
$m=3$,	$\alpha=$		2	3	4	5
	$\epsilon=$	0.00	17.0	6.7	4.2	3.1
		0.01	14.9	6.3	4.0	3.1
		0.05	9.5	5.2	3.6	2.8
		0.10	6.6	4.3	3.2	2.6
$m=25$,	$\epsilon=$	0.00	50.0	13.5	7.1	4.8
		0.01	33.5	12.0	6.7	4.6
		0.05	14.5	8.3	5.4	4.1
		0.10	8.5	6.0	4.4	3.5
$m=100$,	$\epsilon=$	0.00	99.0	21.5	10.0	6.3
		0.01	50.2	17.8	9.2	6.0
		0.05	16.8	10.6	6.9	5.0
		0.10	9.2	7.0	5.3	4.1
$m=250$,	$\epsilon=$	0.00	160	29.1	12.5	7.6
		0.01	61.4	22.8	11.3	7.1
		0.05	17.8	12.1	8.0	5.7
		0.10	9.5	7.6	5.8	4.6
$m=1000$,	$\epsilon=$	0.00	310	46.3	17.7	10.0
		0.01	76.1	31.9	15.2	9.1
		0.05	18.9	14.2	9.7	6.9
		0.10	18.9	14.2	9.7	6.9
		0.10	9.7	8.4	6.6	5.3

The transformation into a straight line relation between ln $(y\text{-}\omega)$ and z of the curvilinear relation between ln$y$ and z reflects an adequate a priori estimate of the parameter ω. However, the effectiveness of this procedure is limited to small and moderate values of the parameter α, since for large values of α the existence of a value $\omega > 0$ is not always clearly reflected.

Presenting Eq. (21) in the normalized form

$$\eta = \epsilon + (1-\epsilon) e^{z/\alpha} \tag{22}$$

where $\eta = y/\beta$ and $\epsilon = \omega/\beta$ so that both parameters ϵ and $1/\alpha$ are bounded between zero and one, it seems that for large values of α the normalized variate η is a nearly linear function of z, although ϵ is not zero. When both α and ϵ are large, the scatter of η is narrow; when both α and ϵ are small, the scatter is wide. Since the effect of either of the two parameters has the same direction, they can compensate each other, a fact that obviously produces problems in the parameter estimation. However, within the range of values of $\alpha < 5$, which is the range of this parameter that is significant for fatigue performance of high-strength aircraft structural metal alloys, the increasing nonlinearity of Eq. (22) with increasing ϵ is quite pronounced and can be used for a first estimation of this parameter.

The distribution function $P(\eta_1)$ of the smallest value η_1 of the normalized variate η in a sample of size m is obtained from

$$1-P(\eta_1) = [1-P(\eta)]^m = \exp\left[-m\left(\frac{\eta_1 - \epsilon}{1-\epsilon}\right)^{\alpha}\right] \tag{23}$$

which is of the same shape as Eq. (20) but with the characteristic value reduced by the factor $m^{-1/\alpha}$. At the reliability level $[1-P(\eta_1)] = R$, therefore

$$\eta_1(R) = \epsilon + (1-\epsilon) m^{-1/\alpha}\left[\ln\left(\frac{1}{R}\right)\right]^{1/\alpha} \tag{24}$$

while the expectation of η according to the theory of extreme values is

$$E(\eta) = \epsilon + (1-\epsilon)\Gamma(1+1/\alpha) \tag{25}$$

The scatter factor at the reliability level R with respect to the expectation $E(\eta)$ is therefore

$$S(R) = S_R = \frac{E(\eta)}{\eta_1(R)} = \frac{\epsilon + (1-\epsilon)\Gamma(1+1/\alpha)}{\epsilon + (1-\epsilon) m^{-1/\alpha}[\ln(1/R)]^{1/\alpha}} \tag{26}$$

At different values of α and reliability levels R it shows the significant effect of this assumption and illustrates the necessity of much more elaborate experimental studies of the magnitude of the minimum fatigue life in different structural metal alloys as well as in different designs for the same metal. The normalized minimum fatigue life ϵ of a whole design or of a design detail might be considered to reflect the quality of design and fabrication more than that of the material. It might therefore provide a quantitative measure of the quality of fatigue design as well as a means of considering such quality in the reliability assessment of a structural detail or a whole structure.

Comparing the results at the selected reliability levels $R=0.5$ to $R=0.99$ for the sample sizes $n=1$ and 3, the practical independence of the figures of the sample size n is most striking. The practical significance of this conclusion is quite obvious; it provides an answer to the question that is always being asked with respect to single full-scale tests: What statistical significance can be associated with their results? The insignificant differences of the computed scatter factors for $n=1$ and $1 < n \leq 3$ establish the result of a single full-scale prototype fatigue test as an adequate basis for the estimation of the probability of survival of *all* members of a fleet of size m at a fraction of the test life specified by the scatter factor.

In this definition of the scatter factor with respect to the expectation of η according to Eq. (26) the estimate of the scale parameter on the basis of a sample of size n is replaced by the expectation of the normalized variable.

The scatter factor as a statistical variable in accordance with Eq. (1), considering, however, the existence of a positive value ϵ in the three-parameter distribution function Eq. (20), can be obtained by introducing the auxiliary variable $S^1 = (\hat{\beta} - \beta)/(y_1 - \omega)$. Solving for $\hat{\beta} = S$, the relation is obtained

$$S = \frac{S^1}{1 + \epsilon(S^1 - 1)} \tag{27}$$

The distribution of the ratio S^1 is given by Eq. (6), since the distributions of variables $x = y - \omega$ with location parameter $v = \beta - \omega$ and $x_1 = y_1 - \omega$ with location parameter $v_1 = (\beta - \omega) m^{1/\alpha}$ have the form of two-parameter distributions of smallest values

$$\ln[1-P(x)] = -\left(\frac{x}{v}\right)^{\alpha} \tag{28}$$

and

$$\ln\left[1-P(x_1)\right] = -\left(\frac{x_1}{v_1}\right)^{\alpha} \tag{29}$$

Applying the rules of transformation of distribution of functions of a random variable[11] the distribution of S is obtained from that of S^1 in the form

$$F(S) = \left\{ \frac{1}{1+\frac{m}{n}\left[\frac{1-S\epsilon}{S(1-\epsilon)}\right]^{\alpha}} \right\}^{n} \tag{30}$$

between the limits $F(S)=0$ at $S=0$ and $F(S)=1$ at $S=\epsilon^{-1}$. For $\epsilon=0$, this distribution degenerates into Eq. (10), and its median

$$\breve{S} = m^{1/\alpha} \frac{1}{[n(2^{1/n}-1)]^{1/\alpha}\left\{1+\epsilon\left[\frac{m}{n({}^{n}(2)^{1/2}-1)}\right]^{1/\alpha}-\epsilon\right\}^{1/\alpha}} \tag{31}$$

degenerates into Eq. (16). Since the reliability R is the probability of values $\leq S$, it follows that $R=F(S)=[F'(S)]^n$ where $F'(S)$ is the expression inside the brackets of Eq. (30); with $n=1$ it represents the distribution of S for a single test.

Equation (30) has been solved for S for different sets of values of the parameters n, m, α and ϵ at different levels of $R=F(S)$; the results are presented in Tables 2-7.

V. Conclusions

Comparison of the values of the scatter factor presented in Tables 2-7 at the three selected reliability levels $R=0.5$, 0.9, and 0.99 leads to the conclusion that for fleet sizes between $m=100$ and $m=250$ the order of magnitude of the currently used conventional scatter factors implies a reliability level in the median range $R\sim0.5$ and a normalized minimum fatigue life of not less than $\epsilon=0.05$. To attain a reliability level $R\geq 0.9$ would require at least doubling the conventional scatter factors, provided a normalized minimum fatigue life of the structure of $\epsilon\leq0.10$ can be assured by built-in redundancies, inspection, and maintenance. Obviously such reliability levels are easier to attain in aluminum and medium-strength steel structures than in titanium and, particularly, in high-strength steel structures.

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